

# Weak Cardinality Theorems for First-Order Logic

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Fundamentals of Computation Theory 2003



# Outline

- 1 History
  - Enumerability in Recursion and Automata Theory
  - Known Weak Cardinality Theorem
  - Why Do Cardinality Theorems Hold Only for Certain Models?
- 2 Unification by First-Order Logic
  - Elementary Definitions
  - Enumerability for First-Order Logic
  - Weak Cardinality Theorems for First-Order Logic
- 3 Applications
  - A Separability Result for First-Order Logic



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# Motivation of Enumerability

## Problem

Many functions are not computable or not efficiently computable.

## Example

- #SAT:  
How many satisfying assignments does a formula have?



# Motivation of Enumerability

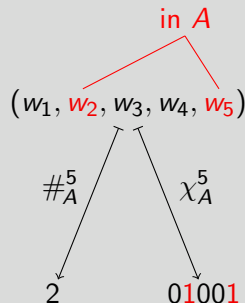
## Problem

Many functions are not computable or not efficiently computable.

## Example

For difficult languages  $A$ :

- Cardinality function  $\#_A^n$ :  
How many input words are in  $A$ ?
- Characteristic function  $\chi_A^n$ :  
Which input words are in  $A$ ?



# Motivation of Enumerability

## Problem

Many functions are not computable or not efficiently computable.

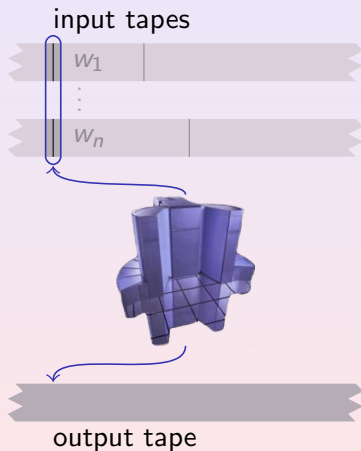
## Solutions

Difficult functions can be

- computed using probabilistic algorithms,
- computed efficiently on average,
- approximated, or
- **enumerated.**



# Enumerators Output Sets of Possible Function Values



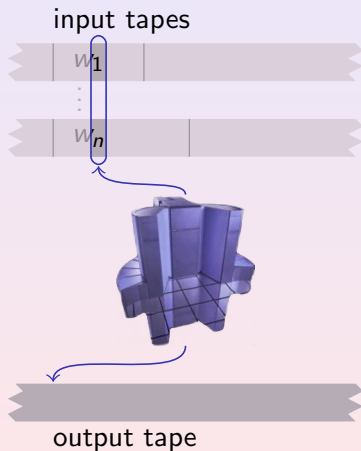
## Definition (1987, 1989, 1994, 2001)

An ***m*-enumerator** for a function  $f$

- ① reads  $n$  input words  $w_1, \dots, w_n$ ,
- ② does a computation,
- ③ outputs at most  $m$  values,
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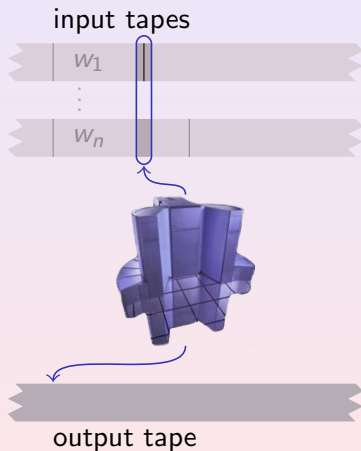
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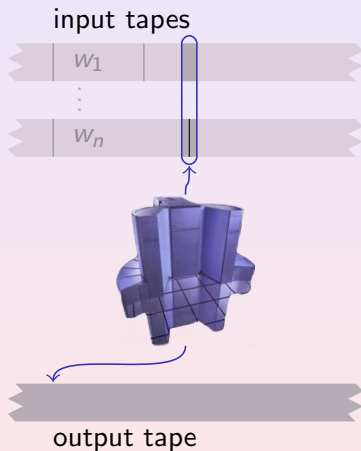


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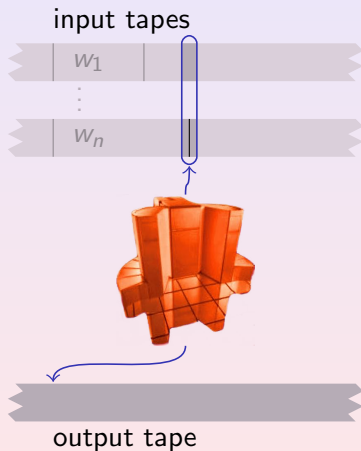
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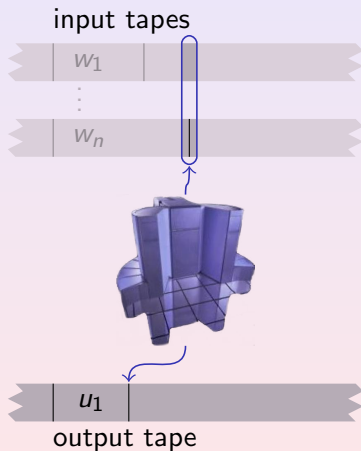
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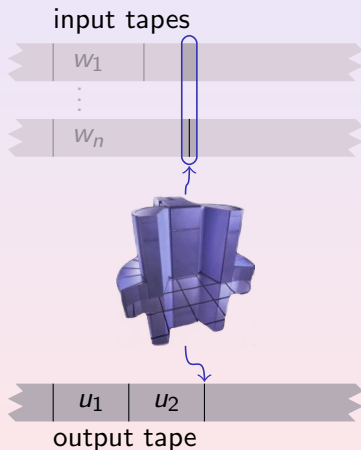
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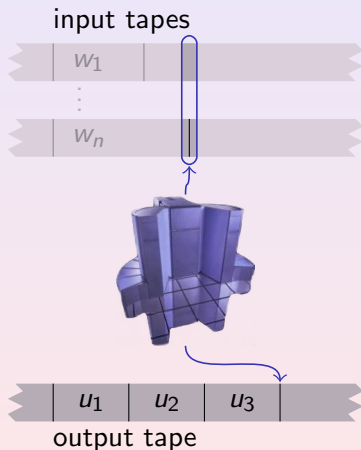


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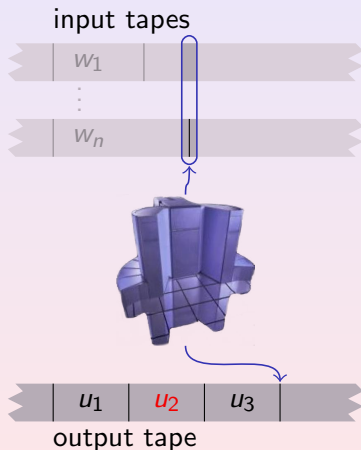
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# How Well Can the Cardinality Function Be Enumerated?

## Observation

For fixed  $n$ , the cardinality function  $\#_A^n$

- can be 1-enumerated by Turing machines only for recursive  $A$ , but
- can be  $(n + 1)$ -enumerated for every language  $A$ .

## Question

What about 2-, 3-, 4-, ...,  $n$ -enumerability?



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# How Well Can the Cardinality Function Be Enumerated by Turing Machines?

## Cardinality Theorem (Kummer, 1992)

*If  $\#_A^n$  is  $n$ -enumerable by a Turing machine, then  $A$  is recursive.*

## Weak Cardinality Theorems (Kummer, 1992)

- 1 *If  $\chi_A^n$  is  $n$ -enumerable by a Turing machine, then  $A$  is recursive.*
- 2 *If  $\#_A^2$  is 2-enumerable by a Turing machine, then  $A$  is recursive.*
- 3 *If  $\#_A^n$  is  $n$ -enumerable by a Turing machine that never enumerates weakly recursive sets, then  $A$  is recursive.*



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# How Well Can the Cardinality Function Be Enumerated by Finite Automata?

## Conjecture

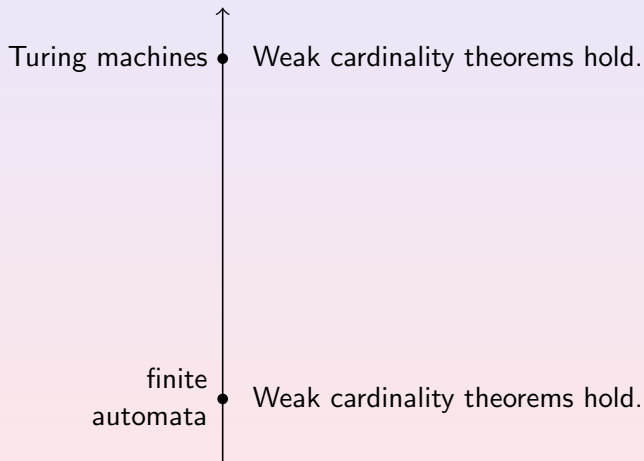
If  $\#_A^n$  is  $n$ -enumerable by a **finite automaton**, then  $A$  is **regular**.

## Weak Cardinality Theorems (2001, 2002)

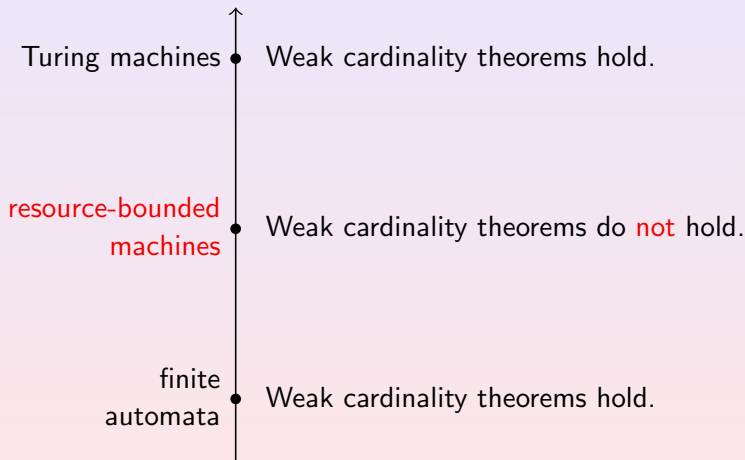
- 1 If  $\chi_A^n$  is  $n$ -enumerable by a **finite automaton**, then  $A$  is **regular**.
- 2 If  $\#_A^2$  is 2-enumerable by a **finite automaton**, then  $A$  is **regular**.
- 3 If  $\#_A^n$  is  $n$ -enumerable by a **finite automaton** that never enumerates both 0 and  $n$ , then  $A$  is **regular**.



# Cardinality Theorems Do Not Hold for All Models



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# Why?

## First Explanation

The weak cardinality theorems hold both for recursion and automata theory **by coincidence**.

## Second Explanation

The weak cardinality theorems hold both for recursion and automata theory, **because they are instantiations of single, unifying theorems**.



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## Second Explanation

The weak cardinality theorems hold both for recursion and automata theory, **because they are instantiations of single, unifying theorems**.

The second explanation is correct.

The theorems can (almost) be unified using first-order logic.



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# What Are Elementary Definitions?

## Definition

A relation  $R$  is **elementarily definable in a logical structure  $\mathcal{S}$**  if

- 1 there exists a first-order formula  $\phi$ ,
- 2 that is true exactly for the elements of  $R$ .

## Example

The set of even numbers is elementarily definable in  $(\mathbb{N}, +)$  via the formula  $\phi(x) \equiv \exists z . z + z = x$ .

## Example

The set of powers of 2 is not elementarily definable in  $(\mathbb{N}, +)$ .



# Characterisation of Classes by Elementary Definitions

## Theorem (Büchi, 1960)

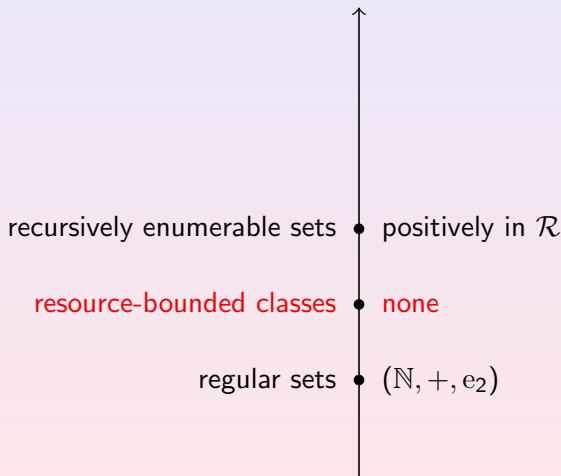
*There exists a logical structure  $(\mathbb{N}, +, e_2)$  such that a set  $A \subseteq \mathbb{N}$  is **regular** iff it is **elementarily definable** in  $(\mathbb{N}, +, e_2)$ .*

## Theorem

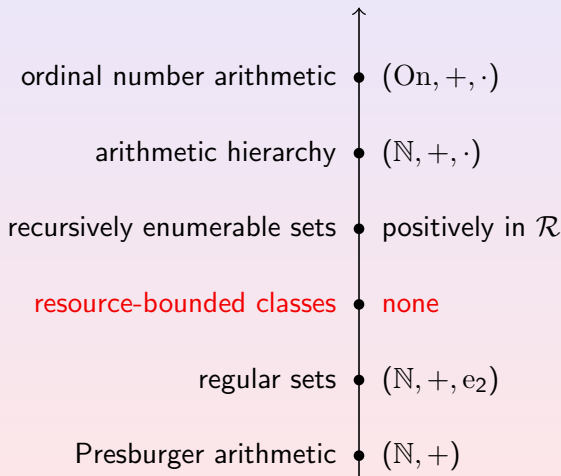
*There exists a logical structure  $\mathcal{R}$  such that a set  $A \subseteq \mathbb{N}$  is **recursively enumerable** iff it is **positively elementarily definable** in  $\mathcal{R}$ .*



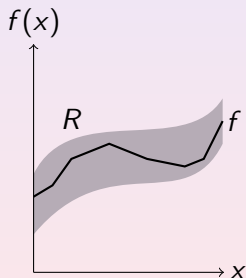
# Characterisation of Classes by Elementary Definitions



# Characterisation of Classes by Elementary Definitions



# Elementary Enumerability is a Generalisation of Elementary Definability



## Definition

A function  $f$  is

elementarily  $m$ -enumerable in a structure  $\mathcal{S}$  if

- ① its graph is contained in an elementarily definable relation  $R$ ,
- ② which is  $m$ -bounded, i.e., for each  $x$  there are at most  $m$  different  $y$  with  $(x, y) \in R$ .

# The Original Notions of Enumerability are Instantiations

## Theorem

*A function is  $m$ -enumerable by a **finite automaton** iff it is elementarily  $m$ -enumerable in  $(\mathbb{N}, +, e_2)$ .*

## Theorem

*A function is  $m$ -enumerable by a **Turing machine** iff it is positively elementarily  $m$ -enumerable in  $\mathcal{R}$ .*



# The First Weak Cardinality Theorem

## Theorem

Let  $S$  be a logical structure with universe  $U$  and let  $A \subseteq U$ . If

- ①  $S$  is well-orderable and
- ②  $\chi_A^n$  is elementarily  $n$ -enumerable in  $S$ ,

then  $A$  is elementarily definable in  $S$ .



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## Corollary

If  $\chi_A^n$  is  $n$ -enumerable by a finite automaton, then  $A$  is regular.



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## Corollary (with more effort)

If  $\chi_A^n$  is  $n$ -enumerable by a Turing machine, then  $A$  is recursive.



# The Second Weak Cardinality Theorem

## Theorem

Let  $S$  be a logical structure with universe  $U$  and let  $A \subseteq U$ . If

- ①  $S$  is well-orderable,
- ② every finite relation on  $U$  is elementarily definable in  $S$ , and
- ③  $\#_A^2$  is elementarily 2-enumerable in  $S$ ,

then  $A$  is elementarily definable in  $S$ .



# The Third Weak Cardinality Theorem

## Theorem

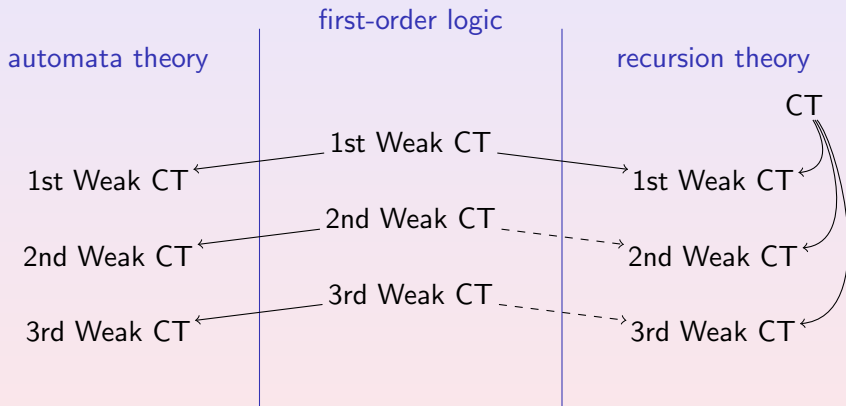
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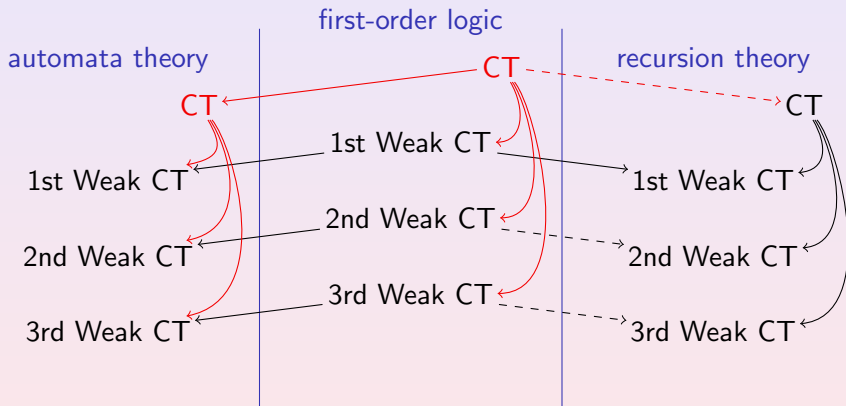
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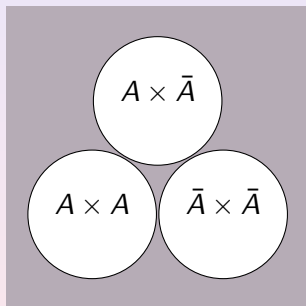
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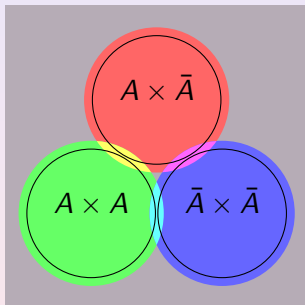
*Let  $S$  be a well-orderable logical structure in which all finite relations are elementarily definable.*

*If there exist elementarily definable supersets of  $A \times A$ ,  $A \times \bar{A}$ , and  $\bar{A} \times \bar{A}$  whose intersection is empty, then  $A$  is elementarily definable in  $S$ .*

## Note

The theorem is no longer true if we add  $\bar{A} \times A$  to the list.





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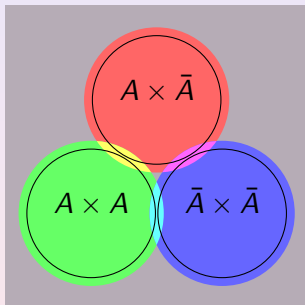
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# Summary

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- The weak cardinality theorems for first-order logic **unify** the weak cardinality theorems of automata and recursion theory.
- The logical approach yields weak cardinality theorems for **other computational models**.
- Cardinality theorems are **separability theorems** in disguise.

## Open Problems

- Does a cardinality theorem for first-order logic hold?
- What about non-well-orderable structures like  $(\mathbb{R}, +, \cdot)$ ?

